MODIFIED EQUINOCTIAL ORBITAL ELEMENTS

The modified equinoctial orbital elements are a set of orbital elements that are useful for trajectory analysis and optimization. They are valid for circular, elliptic, and hyperbolic orbits. These *direct* modified equinoctial equations exhibit no singularity for zero eccentricity and orbital inclinations equal to 0 and 90 degrees. However, two of the components are singular for an orbital inclination of 180 degrees.

Relationship between modified equinoctial and classical orbital elements

$$
\begin{aligned}
p &= a\left(1-e^2\right) \\
f &= e\cos\left(\omega+\Omega\right) \\
g &= e\sin\left(\omega+\Omega\right) \\
h &= \tan\left(i/2\right)\cos\Omega \\
k &= \tan\left(i/2\right)\sin\Omega \\
L &= \Omega+\omega+\theta
\end{aligned}
\tag{1}
$$

where

$p =$ semiparameter
$a =$ semimajor axis
$e =$ orbital eccentricity
$i =$ orbital inclination
$\omega =$ argument of perigee
$\Omega =$ right ascension of the ascending node
$\theta =$ true anomaly
$L =$ true longitude

Relationship between classical and modified equinoctial orbital elements

semimajor axis

$$a = \frac{p}{1-f^2-g^2} \tag{2a}$$

orbital eccentricity

$$e = \sqrt{f^2+g^2} \tag{2b}$$

orbital inclination

$$i = 2\tan^{-1}\left(\sqrt{h^2+k^2}\right) = \tan^{-1}\left(2\sqrt{h^2+k^2}, 1-h^2-k^2\right) \tag{2c}$$

argument of periapsis

$$\omega = \tan^{-1}\left(g/f\right) - \tan^{-1}\left(k/h\right) = \tan^{-1}\left(gh - fk, fh + gk\right) \tag{2d}$$

right ascension of the ascending node

$$\Omega = \tan^{-1}\left(k, h\right) \tag{2e}$$

true anomaly

$$\theta = L - \left(\Omega + \omega\right) = L - \tan^{-1}\left(g/f\right) \tag{2f}$$

argument of latitude

$$u = \omega + \theta = \tan^{-1}\left(h\sin L - k\cos L, h\cos L + k\sin L\right) \tag{2g}$$

In these equations an expression of the form $\tan^{-1}\left(a,b\right)$ indicates a four quadrant inverse tangent calculation.

Relationship between ECI state vector and modified equinoctial elements

position vector

$$\mathbf{r} = \begin{bmatrix} \dfrac{r}{s^2}\left(\cos L + \alpha^2 \cos L + 2hk \sin L\right) \\ \dfrac{r}{s^2}\left(\sin L - \alpha^2 \sin L + 2hk \cos L\right) \\ \dfrac{2r}{s^2}\left(h\sin L - k\cos L\right) \end{bmatrix} \tag{3a}$$

velocity vector

$$\mathbf{v} = \begin{bmatrix} -\dfrac{1}{s^2}\sqrt{\dfrac{\mu}{p}}\left(\sin L + \alpha^2 \sin L - 2hk\cos L + g - 2fhk + \alpha^2 g\right) \\ -\dfrac{1}{s^2}\sqrt{\dfrac{\mu}{p}}\left(-\cos L + \alpha^2 \cos L + 2hk\sin L - f + 2ghk + \alpha^2 f\right) \\ \dfrac{2}{s^2}\sqrt{\dfrac{\mu}{p}}\left(h\cos L + k\sin L + fh + gk\right) \end{bmatrix} \tag{3b}$$

where

$$\alpha^2 = h^2 - k^2$$

$$s^2 = 1 + h^2 + k^2$$

$$r = \frac{p}{w}$$

$$w = 1 + f\cos L + g\sin L$$

Modified equinoctial form of the orbital equations of motion

The system of *first-order* modified equinoctial equations of orbital motion are given by the following expressions

$$\dot{p} = \frac{dp}{dt} = \frac{2p}{w}\sqrt{\frac{p}{\mu}}\Delta_t$$

$$\dot{f} = \frac{df}{dt} = \sqrt{\frac{p}{\mu}}\left[\Delta_r \sin L + \left[\left(w+1\right)\cos L + f\right]\frac{\Delta_t}{w} - \left(h\sin L - k\cos L\right)\frac{g\Delta_n}{w}\right]$$

$$\dot{g} = \frac{dg}{dt} = \sqrt{\frac{p}{\mu}}\left[-\Delta_r \cos L + \left[\left(w+1\right)\sin L + g\right]\frac{\Delta_t}{w} + \left(h\sin L - k\cos L\right)\frac{g\Delta_n}{w}\right]$$

$$\dot{h} = \frac{dh}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2\Delta_n}{2w}\cos L$$

$$\dot{k} = \frac{dk}{dt} = \sqrt{\frac{p}{\mu}}\frac{s^2\Delta_n}{2w}\sin L$$

$$\dot{L} = \frac{dL}{dt} = \sqrt{\mu p}\left(\frac{w}{p}\right)^2 + \frac{1}{w}\sqrt{\frac{p}{\mu}}\left(h\sin L - k\cos L\right)\Delta_n \tag{4}$$

where $\Delta_r, \Delta_t, \Delta_n$ are *non-two-body* perturbations in the radial, tangential and normal directions, respectively. The radial direction is along the geocentric radius vector of the spacecraft measured positive in a direction away from the geocenter, the tangential direction is perpendicular to this radius vector measured positive in the direction of orbital motion, and the normal direction is positive along the angular momentum vector of the spacecraft's orbit.

In vector form the equations of motion can be expressed as follows:

$$\dot{\mathbf{y}} = \frac{d\mathbf{y}}{dt} = \mathbf{A}\left(\mathbf{y}\right)\mathbf{P} + \mathbf{b} \tag{5}$$

where

$$\mathbf{A} = \left\{ \begin{array}{ccc} 0 & \dfrac{2p}{w}\sqrt{\dfrac{p}{\mu}} & 0 \\ \sqrt{\dfrac{p}{\mu}}\sin L & \sqrt{\dfrac{p}{\mu}}\dfrac{1}{w}\left[\left(w+1\right)\cos L + f\right] & -\sqrt{\dfrac{p}{\mu}}\dfrac{g}{w}\left[h\sin L - k\cos L\right] \\ -\sqrt{\dfrac{p}{\mu}}\cos L & \sqrt{\dfrac{p}{\mu}}\left[\left(w+1\right)\sin L + g\right] & \sqrt{\dfrac{p}{\mu}}\dfrac{f}{w}\left[h\sin L - k\cos L\right] \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2\cos L}{2w} \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\dfrac{s^2\sin L}{2w} \\ 0 & 0 & \sqrt{\dfrac{p}{\mu}}\left[h\sin L - k\cos L\right] \end{array} \right\}$$

and

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{\mu p}\left(\dfrac{w}{p}\right)^2 \end{bmatrix}^T$$

The total non-two-body acceleration vector is given by

$$\mathbf{P} = \Delta_r \hat{\mathbf{i}}_r + \Delta_t \hat{\mathbf{i}}_t + \Delta_n \hat{\mathbf{i}}_n \tag{6}$$

where $\hat{\mathbf{i}}_r, \hat{\mathbf{i}}_t$ and $\hat{\mathbf{i}}_n$ are unit vectors in the radial, tangential and normal directions computed from the ECI position **r** and velocity vectors **v** according to

$$\begin{aligned} \hat{\mathbf{i}}_r &= \frac{\mathbf{r}}{\left\|\mathbf{r}\right\|} \\ \hat{\mathbf{i}}_n &= \frac{\mathbf{r}\times\mathbf{v}}{\left\|\mathbf{r}\times\mathbf{v}\right\|} \\ \hat{\mathbf{i}}_t &= \hat{\mathbf{i}}_n \times \hat{\mathbf{i}}_r = \frac{\left(\mathbf{r}\times\mathbf{v}\right)\times\mathbf{r}}{\left\|\mathbf{r}\times\mathbf{v}\right\|\left\|\mathbf{r}\right\|} \end{aligned} \tag{7}$$

For *unperturbed* two-body motion, $\mathbf{P} = 0$ and the first five equations of motion are simply $\dot{p} = \dot{f} = \dot{g} = \dot{h} = \dot{k} = 0$. Therefore, for two-body motion these modified equinoctial orbital elements are constant.

Non-spherical Earth Gravity

The non-spherical gravitational acceleration vector can be expressed as

$$\mathbf{g} = g_N \, \hat{\mathbf{i}}_N - g_r \, \hat{\mathbf{i}}_r \tag{8}$$

where

$$\hat{\mathbf{i}}_N = \frac{\hat{\mathbf{e}}_N - \left(\hat{\mathbf{e}}_N^T \hat{\mathbf{i}}_r\right)\hat{\mathbf{i}}_r}{\left\| \hat{\mathbf{e}}_N - \left(\hat{\mathbf{e}}_N^T \hat{\mathbf{i}}_r\right)\hat{\mathbf{i}}_r \right\|}$$

and

$$\hat{\mathbf{e}}_N = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

In these equations the north direction component is indicated by subscript *N* and the radial direction component is subscript *r*.

The contributions due to the *zonal* gravity effects of J_2, J_3, J_4 are as follows:

$$g_N = -\frac{\mu \cos\phi}{r^2} \sum_{k=2}^{4} \left(\frac{R_e}{r}\right)^k P_k' J_k \tag{9a}$$

$$g_r = -\frac{\mu}{r^2} \sum_{k=2}^{4} \left(k+1\right) \left(\frac{R_e}{r}\right)^k P_k J_k \tag{9b}$$

where

$\mu =$ gravitational constant
$r =$ geocentric distance of the spacecraft
$R_e =$ equatorial radius of the Earth
$\phi =$ geocentric latitude
$J_k =$ zonal gravity coefficient
$P_k =$ k^{th} order Legendre polynomial

For a zonal only Earth gravity model, the east component is identically zero.

Therefore, the zonal gravity perturbation contribution is

$$\mathbf{a}_g = \mathbf{Q}^T \mathbf{g} \tag{10}$$

where $\mathbf{Q} = \left[\hat{\mathbf{i}}_r \quad \hat{\mathbf{i}}_t \quad \hat{\mathbf{i}}_n \right]$.

For J_2 effects only, the components are as follows:

$$\Delta_{J_{2_r}} = -\frac{3\mu J_2 R_e^2}{2r^4}\left[1 - \frac{12\left(h\sin L - k\cos L\right)^2}{\left(1+h^2+k^2\right)^2}\right] \tag{11a}$$

$$\Delta_{J_{2_t}} = -\frac{12\mu J_2 R_e^2}{r^4}\left[\frac{\left(h\sin L - k\cos L\right)\left(h\cos L + k\sin L\right)}{\left(1+h^2+k^2\right)^2}\right] \tag{11b}$$

$$\Delta_{J_{2_n}} = -\frac{6\mu J_2 R_e^2}{r^4}\left[\frac{\left(1-h^2-k^2\right)\left(h\sin L - k\cos L\right)}{\left(1+h^2+k^2\right)^2}\right] \tag{11c}$$

Aerodynamic Drag

The radial, tangential and normal perturbations due to aerodynamic drag are given by the following three expressions:

$$\Delta_{D_r} = -\frac{1}{2}\rho S C_D v v_r \tag{12a}$$

$$\Delta_{D_t} = -\frac{1}{2}\rho S C_D v v_t \tag{12b}$$

$$\Delta_{D_n = 0} \tag{12c}$$

where

$$\begin{aligned}
\rho &= \text{ atmospheric density} \\
S &= \text{ aerodynamic reference area} \\
C_D &= \text{ drag coefficient} \\
v &= \text{ velocity magnitude} \\
v_r &= \sqrt{\frac{\mu}{p}}\left(f\sin L - g\cos L\right) \\
v_t &= \sqrt{\frac{\mu}{p}}\left(1 + f\cos L + g\sin L\right)
\end{aligned}$$

Secondary Body Perturbations

The general vector equation for secondary body perturbations such as the Moon or planets is given by

$$\mathbf{t} = -\sum_{j=1}^{n} \mu_j \left[\frac{\mathbf{d}_j}{d_j^3} + \frac{\mathbf{s}_j}{s_j^3} \right] \tag{13}$$

In this equation, $\mathbf{s}_j$ is the vector from the primary body to the secondary body j, μ_j is the gravitational constant of the secondary body and $\mathbf{d}_j = \mathbf{r} - \mathbf{s}_j$, where $\mathbf{r}$ is the position vector of the spacecraft relative to the primary body.

To avoid numerical problems, use is made of Battin's $F(q)$ function given by

$$F(q_k) = q_k \left[\frac{3 + 3q_k + q_k^2}{1 + \left(\sqrt{1 + q_k}\right)^3} \right] \tag{14}$$

where

$$q_k = \frac{\mathbf{r}^T (\mathbf{r} - 2\mathbf{s}_k)}{\mathbf{s}_k^T \mathbf{s}_k}$$

The $\mathbf{t}$ term can now be expressed as

$$\mathbf{t} = -\sum_{k=1}^{n} \frac{\mu_k}{d_k^3} \left[\mathbf{r} + F(q_k) \mathbf{s}_k \right] \tag{15}$$

Finally, the perturbation due to secondary bodies in the modified equinoctial coordinate system is given by

$$\mathbf{a}_{TB} = \mathbf{Q}^T \mathbf{t} \tag{16}$$

where $\mathbf{Q} = \left[\hat{\mathbf{i}}_r \quad \hat{\mathbf{i}}_t \quad \hat{\mathbf{i}}_n \right]$.

Propulsive Thrust

The acceleration due to propulsive thrust can be expressed as

$$\mathbf{a}_T = \frac{T}{m} \hat{\mathbf{u}} \tag{17}$$

where T is the thrust, m is the spacecraft mass and $\hat{\mathbf{u}} = \left[u_r \quad u_t \quad u_n \right]$ is the unit pointing thrust vector expressed in the spacecraft-centered radial-tangential-normal coordinate system. The components of the unit thrust vector can also be defined in terms of the in-plane pitch angle θ and the out-of-plane yaw angle ψ as follows:

$$
\begin{aligned}
u_r &= \sin\theta \\
u_t &= \cos\theta\cos\psi \\
u_n &= \cos\theta\sin\psi
\end{aligned}
\tag{18}
$$

Finally, the pitch and yaw angles can be determined from the components of the unit thrust vector according to

$$
\begin{aligned}
\theta &= \sin^{-1}\left(u_r\right) \\
\psi &= \tan^{-1}\left(u_n, u_t\right)
\end{aligned}
\tag{19}
$$

The pitch angle is positive above the "local horizontal" and the yaw angle is positive in the direction of the angular momentum vector.

The relationship between a unit thrust vector in the ECI coordinate system $\hat{\mathbf{u}}_{T_{ECI}}$ and the corresponding unit thrust vector in the modified equinoctial system $\hat{\mathbf{u}}_{T_{MEE}}$ is given by

$$
\hat{\mathbf{u}}_{T_{ECI}} = \left[\hat{\mathbf{i}}_r \quad \hat{\mathbf{i}}_t \quad \hat{\mathbf{i}}_n\right]\hat{\mathbf{u}}_{T_{MEE}} \tag{20}
$$

where

$$
\hat{\mathbf{i}}_r = \frac{\mathbf{r}}{|\mathbf{r}|} \qquad \hat{\mathbf{i}}_n = \frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|} \qquad \hat{\mathbf{i}}_t = \hat{\mathbf{i}}_n \times \hat{\mathbf{i}}_r = \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{|\mathbf{r}\times\mathbf{v}||\mathbf{r}|}
$$

This relationship can also be expressed as

$$
\hat{\mathbf{u}}_{T_{ECI}} = [Q]\hat{\mathbf{u}}_{T_{MEE}} = \begin{bmatrix} \hat{\mathbf{r}}_x & \left(\hat{\mathbf{h}}\times\hat{\mathbf{r}}\right)_x & \hat{\mathbf{h}}_x \\ \hat{\mathbf{r}}_y & \left(\hat{\mathbf{h}}\times\hat{\mathbf{r}}\right)_y & \hat{\mathbf{h}}_y \\ \hat{\mathbf{r}}_z & \left(\hat{\mathbf{h}}\times\hat{\mathbf{r}}\right)_z & \hat{\mathbf{h}}_z \end{bmatrix}\hat{\mathbf{u}}_{T_{MEE}} \tag{21}
$$

Finally, the transformation of the unit thrust vector in the ECI system to the modified equinoctial coordinate system is given by

$$
\hat{\mathbf{u}}_{T_{MEE}} = [Q]^T\,\hat{\mathbf{u}}_{T_{ECI}} \tag{22}
$$

For the case of tangential steering

$$
\hat{\mathbf{u}}_{T_{ECI}} = \left[\left(\hat{\mathbf{h}}\times\hat{\mathbf{r}}\right)_x \quad \left(\hat{\mathbf{h}}\times\hat{\mathbf{r}}\right)_y \quad \left(\hat{\mathbf{h}}\times\hat{\mathbf{r}}\right)_z\right]^T \tag{23}
$$

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